ON OSCILLATIONS OF AN AXIALLY TRANSLATING TENSIONED STRING-LIKE EQUATION UNDER INTERNAL DAMPING

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ABSTRACT: *This research article studies an internally damped linear homogeneous string-like equation. Both ends of the string are held fixed, whereas general initial displacement and velocity are considered. From mechanical and physical aspects the problem describes a mathematical model of internally damped transversal vibrations of a moving or an elastic drive. From Hamilton's principle, a second order partial differential equation (PDE) for axially moving continuum is formulated. The axial speed of string is considered to be positive, constant and small compared to wave velocity, and it is also assumed that the introduced internal damping is small. The solutions of equation of motion are based upon two timescales method. By application of this method, it will be shown that the internal damping does in fact affect the solution responses, and reduces the vibration and noise in the system. It will also be shown that the damping generated in the belt system depends on the mode number n, which is obviously expected from mechanical point of view.*

Keywords: Conveyor belt, String-like, Axially moving, Internal damping, Two timescales

INTRODUCTION

Physical and mechanical systems are generally oscillatory systems. Axially translating systems, for example, are appearing in such category. Axially translating systems have received much research attention since last six decades. Axially translating systems have been observed in many practical and engineering situations. The energy dissipation, known as damping, can easily be associated to axially translating systems, see Refs. [1-3]. Axially translating systems have many engineering applications. For example, conveyor belt systems, as given in Refs. [4-6], magnetic tapes, pipes conveying fluids, data saving devices, and elevator cable systems, see Ref. [5], and all such kind of systems are bound to vibrations.

The research study of axially translating systems with constant or time-varying velocity and viscous, internal or boundary damping have received much importance in manufacture and design. It is common experience that the vibration causes severe failures to many mechanical or physical structures. In this view, it becomes necessary to design systems where unnecessary noise and vibrations can be reduced by means of solid procedures. Tacoma Narrows bridge has been a good example in education and research institutions for a structural collapse. This collapse was due to winds at certain speed. Apart from damaging structures, the vibration also causes human problems, that is, vibrations create anxiety to society. Keeping in view the effects of vibrations, it is matter of necessity to formulate methods and procedures to decrease vibrations from the physical and mechanical systems.

In many cases, damping devices can be kept through the support conditions to control vibrations through boundaries as seen in Refs. [4-7]. In Ref. [8] the damping device is attached through whole spatial domain of the translating system. The reflection and damping properties for a wave equation have been studied in Ref. [9], where the authors have provided interesting results for a semi-infinite string. For different boundary conditions, in Ref. [10], the authors have provided detailed analysis for the energetic of the elevator cable systems. The authors, in Ref. [11], studied the energetics of an axially translating continuum. They studied the case for fixed supports for string-like problem and the case for simple supports in case of beam-like problem. In Ref. [12], authors have provided analysis of dampers connected at middle of string and beam. But the position of the damper plays significant role. If the damper is introduced at wrong spatial position it may increase energy of motion and may destabilize the system, for details see Ref. [13].

In this article, an internally damped string-like equation is considered. The article has been organized in following way. In Section 2 the governing equations of motions are formulated systematically from the physical principles. In Section 3 the analytic approximations of solutions of an initial boundary value problem are obtained by using multiple timescales method. Section 4 discusses results obtained in Section 4, and Section 5 represents the concluding remarks and future research directions.

THE GOVERNING EQUATIONS

This section is based on equations of motion related to axially translating elastic system with suitable initial and the boundary conditions will be formulated by the application of the Hamilton's principle, see for instance, Ref. [8]. Consider a conveyor belt system which moves with an axial velocity $\bar{V}$ between a pair of pulleys that are located at a distance L meters apart. The transversal vibrations of the conveyor belt system can be modeled as a tensioned string-like equation. The mathematical model of a traveling tensioned string under internal damping is based on the following assumptions:

- The variable x is the spatial coordinate, the variable t is the time,
- $u(x,t)$ models the displacement field in vertical direction from equilibrium position,
- $\frac{Du}{Dt} = u_t(x,t) + \bar{V}u_x(x,t)$ is the material velocity, where $u_t(x,t)$ is the local velocity at a fixed position at time t and $\bar{V}u_x(x,t)$ is transversal velocity component due to axial velocity $\bar{V}$ with slope $u_x(x,t)$ at a fixed time t,
- $P = P(x,t)$, is component of tension in vertical direction,
- The mass of string per unit length, ρ, is constant,
- The internal damping coefficient $\bar{\beta}$ is assumed to be constant,

- The transversal vibrations are assumed to be small, and
- The effects of the gravity and the other forces are neglected.

To be more specific about the vertical component of the tension $P = P(x,t)$, it is possible to include the material damping in the system. It is evident that during the vibration of the string in the vertical direction particles are constantly rubbing each other as the string vibrates. This process of rubbing of the particles converts some kinetic energy into heat and so decreases the tension in the string and damp out the vibrations. As the vibrations become faster, the more heat is generated in the system. Thus, the tension in the string, $P = P(x,t)$, does not only depend on the relative displacements $u_x(x,t)$, but also on the total time rate of change of these relative displacements, $\frac{D}{Dt}(u_x) = u_{xt} + \bar{V}u_{xx}$, since the string moves with velocity $\bar{V}$ in positive x direction. Thus, the total vertical component of the tension becomes $P(x,t) = Tu_{xx}(x,t) + \bar{\beta}(u_{xxt}(x,t) + \bar{V}u_{xxx}(x,t))$, where T is the constant horizontal tension in the string and where $\bar{\beta}$ is the coefficient of the material damping which is assumed to be constant. Thus, the string-like equation is given as follows,

$$\rho\left(u_{tt} + 2\bar{V}u_{xt} + \dot{\bar{V}}u_x + \bar{V}^2u_{xx}\right) - Tu_{xx} - \bar{\beta}(u_{xxt} + \bar{V}u_{xxx}) = 0;\ 0 < x < L, t > 0, \quad (1)$$

with the fixed boundary conditions,

$$u(0,t) = u(L,t) = 0; \quad t > 0 \quad (2)$$

and the general initial conditions,

$$u(x,0) = f(x), \text{and } u_t(x,0) = g(x); \quad (3)$$

The terms in the bracket into Eq. (1) represent acceleration quantities. The equations contained in (1)-(3) can be put into a non-dimensional form by using following dimensionless quantities: $u^* = \frac{u}{L}, x^* = \frac{x}{L}, t^* = \frac{ct}{L}, V_0 = \frac{\bar{V}}{c}, \delta_0 = \frac{\bar{\beta}}{\rho c L}, f^* = \frac{f}{L}, g^* = \frac{g}{c}$, where $c = \sqrt{T/\rho}$ is the wave velocity. Thus, the equations (1)-(3) into non-dimensional form become:

$$u_{tt} + 2V_0u_{xt} + \dot{V}_0u_x + (V_0^2 - 1)u_{xx} - \delta_0(u_{xxt} + V_0u_{xxx}) = 0. \quad (4)$$

The boundary conditions are given as,

$$u(0,t) = 0, \text{and } u(1,t) = 0. \quad (5)$$

The initial conditions are given as,

$$u(x,0) = f(x), \text{and } u_t(x,0) = g(x). \quad (6)$$

THE ANALYTICAL APPROXIMATION

This section is devoted to construction of an approximation of the solutions to the initial-boundary value problem (4)-(6) by using a multiple timescales perturbation method. For a complete overview of this method, see Refs. [14,15,16]. Following two assumptions are made to utilize a two timescales perturbation method. The axial velocity $\bar{V}$ of the string is assumed to be small compared to wave velocity c and that the damping coefficient $\bar{\beta}$ is small compared to ρcL. Based on these two assumptions, it is reasonable to write $V_0 = \frac{\bar{V}}{c} = O(\varepsilon)$, and $\delta_0 = \frac{\bar{\beta}}{\rho cL} = O(\varepsilon)$, that is, $V_0 = \varepsilon V$ and $\delta_0 = \varepsilon\delta$. The parameter ε is a dimensionless small parameter as described by $0 < \varepsilon \ll 1$. Utilizing these assumptions in Eqs. (4)-(6), it follows that

$$u_{tt} - u_{xx} = -2\varepsilon Vu_{xt} - \varepsilon^2V^2u_{xx} + \varepsilon\delta u_{xxt} + \varepsilon^2\delta Vu_{xxx}, \quad (7)$$

$$u(0,t;\varepsilon) = 0, \text{and } u(1,t;\varepsilon) = 0, \quad (8)$$

$$u(x,0;\varepsilon) = f(x), \text{ and } u_t(x,0;\varepsilon) = g(x). \quad (9)$$

According to a two timescales method a function $u(x,t;\varepsilon)$ is supposed to be a function of spatial variable *x*, the fast timescale *t=t* and, the slow timescale *τ=εt*. For this reason,

$$u(x,t;\varepsilon) = y(x,t,\tau;\varepsilon). \quad (10)$$

By using Eq. (10), the time derivatives can be transformed as follows,

$$u_t = y_t + \varepsilon y_\tau, \quad u_{tt} = y_{tt} + 2\varepsilon y_{t\tau} + \varepsilon^2 y_{\tau\tau}. \quad (11)$$

By substituting Eqs. (10)-(11) into Eqs. (7)-(9), the problem in y up to $O(\varepsilon)$ is given as follows,

$$\begin{gathered} y_{tt} - y_{xx} = -2\varepsilon y_{t\tau} - 2\varepsilon Vy_{xt} + \varepsilon\delta y_{xxt}, \\ y(0,t,\tau;\varepsilon) = y(1,t,\tau;\varepsilon) = 0, \\ y(x,0,0;\varepsilon) = f(x), y_t(x,0,0;\varepsilon) = g(x) - \varepsilon y_\tau(x,0,0;\varepsilon). \end{gathered} \quad (12)$$

Usually it is assumed that not only the function $u(x,t;\varepsilon)$ can be approximated by the asymptotic expansion, but also the function $u(x,t;\varepsilon) = y(x,t,\tau;\varepsilon)$ can be approximated in the powers of ε in the asymptotic expansion as follows,

$$y(x,t,\tau;\varepsilon) = y_0(x,t,\tau) + \varepsilon y_1(x,t,\tau) + \varepsilon^2\cdots, \quad (13)$$

and that all the y_j's for $j = 0,1,2,3\cdots$, are found in such a way that no unbounded (secular) terms arise. It is also assumed that the unknown functions y_j are $O(1)$. Now, by substituting Eq. (13) and its subsequent derivatives into Eq. (12), then by equating the powers of ε^0 and ε^1, and neglecting the ε^2 and the higher powers of ε, the $O(1)$-problem is followed as given by,

$$\begin{gathered} {y_0}_{tt} - {y_0}_{xx} = 0, \\ y_0(0,t,\tau) = y_0(1,t,\tau) = 0, \\ y_0(x,0,0) = f(x), {y_0}_t(x,0,0) = g(x). \end{gathered} \quad (14)$$

The $O(\varepsilon)$-problem is given as,

$$\begin{gathered} {y_1}_{tt} - {y_1}_{xx} = -2{y_0}_{t\tau} - 2V{y_0}_{xt} + \delta{y_0}_{xxt}, \\ y_1(0,t,\tau) = y_1(0,t,\tau) = 0, \\ y_1(x,0,0) = 0, {y_1}_t(x,0,0) = -{y_0}_\tau(x,0,0). \end{gathered} \quad (15)$$

It can be observed that the $O(1)$-problem has solution only for the positive eigenvalues, $\lambda = (n\pi)^2, n \in \mathbb{N}$, for details the reader is referred to Ref. [17,18]. Thus, the solution of $O(1)$-problem is given as follows,

$$y_0(x,t,\tau) = \sum_{n=1}^{\infty}(A_{n0}(\tau)\cos(n\pi t) + B_{n0}(\tau)\sin(n\pi t))\sin(n\pi x), \quad (16)$$

where A_{n0} and B_{n0} are undetermined functions of slow variable τ, often known as Fourier coefficients and, they can be obtained from the $O(\varepsilon)$-problem. The values of the constants $A_{n0}(0)$and $B_{n0}(0)$ can easily be obtained by the initial values as given in Eq. (14) and by using the orthogonality properties of the eigenfunctions. The eigenfunctions $\sin(p\pi x)\, p \in \mathbb{N}$ satisfy the following orthogonality properties, as given by

$$\int_0^1 \sin(p\pi x)\sin(q\pi x)dx = 0, \text{for } p \neq q \\ = \frac{1}{2}, \text{for } p = q. \tag{17}$$

Thus, by using the initial values as given in Eq. (14) and the orthogonality properties of the eigenfunctions as given in Eq. (17), A_{n0} and B_{n0} are given as

$$A_{n0}(0) = 2\int_0^1 f(x)\sin(n\pi x)\,dx, \tag{18}$$

$$n\pi B_{n0}(0) = 2\int_0^1 g(x)\sin(n\pi x)dx. \tag{19}$$

Now, the eigenfunction expansion method is introduced in solving the $O(\varepsilon)$-problem. Following form for the solution $y_1(x,t,\tau)$ is assumed,

$$y_1(x,t,\tau) = \sum_{n=1}^{\infty} w_n(t,\tau)\phi_n(x), \tag{20}$$

where $w_n(t,\tau)$ are the unknown functions of t and τ and are called generalized Fourier coefficients, and where $\phi_n(x) = \sin(n\pi x)$ are the eigenfunctions. Thus, by substitution of Eq. (20) in the $O(\varepsilon)$-equation, it yields

$$\sum_{n=1}^{\infty}\Big(w_{n_{tt}}(t,\tau) + (n\pi)^2 w_n(t,\tau)\Big)\phi_n(x) \\ = -2y_{0_{t\tau}} - 2Vy_{0_{xt}} + \delta y_{0_{xxt}}. \tag{21}$$

Now, by substitution of the solution $y_0(x,t,\tau)$ from Eq. (16) into Eq. (21), it follows that

$$\sum_{n=1}^{\infty}\Big(w_{n_{tt}}(t,\tau) + (n\pi)^2 w_n(t,\tau)\Big)\phi_n(x) \\ = -2\sum_{n=1}^{\infty} R_{n_{t\tau}}(t,\tau)\,\phi_n(x) \\ - 2V\sum_{n=1}^{\infty} R_{n_t}(t,\tau)\,\phi_n'(x) \\ + \delta\sum_{n=1}^{\infty} R_{n_t}(t,\tau)\,\phi_n''(x), \tag{22}$$

where $R_n(t,\tau)$ is given by,

$$R_n(t,\tau) = A_{n0}(\tau)\cos(n\pi t) + B_{n0}(\tau)\sin(n\pi t). \tag{23}$$

By multiplying both sides of Eq. (22) with $\phi_m(x)$, then by integrating the so-obtained equation from $x = 0$ to $x = 1$ with application of the orthogonality property of eigenfunctions, it follows that

$$w_{m_{tt}}(t,\tau) + (m\pi)^2 w_m(t,\tau) \\ = -2R_{m_{t\tau}} - 4VR_{m_t}\Theta_{mm} \\ - \delta(m\pi)^2 R_{m_t} \\ - 4V\sum_{n=1,n\neq m}^{\infty} R_{n_t}\Theta_{nm}, \tag{24}$$

where Θ_{nm} are constants depending on the indices m and n, and their values are given as follows

$$\Theta_{nm} = \int_0^1 \phi_n'(x)\phi_m(x)dx. \tag{25}$$

Note that $\Theta_{mm} = 0$ for all integers m. Thus, by using this value of Θ_{mm}, changing index from m to n and making use of Eq. (23) into Eq. (24), it readily follows that

$$w_{n_{tt}} + (n\pi)^2 w_n \\ = n\pi\big(2A_{n0}'(\tau) + (n\pi)^2\delta A_{n0}(\tau)\big)\sin(n\pi t) \\ - n\pi\big(2B_{n0}'(\tau) + (n\pi)^2\delta B_{n0}(\tau)\big)\cos(n\pi t) \\ + \sum_{m=1,m\neq n}^{\infty} 4V(m\pi)^2\,(A_{m0}(\tau)\sin(m\pi t) \\ - B_{m0}(\tau)\cos(m\pi t))\Theta_{mn} \tag{26}$$

On right hand side of Eq. (26) first two terms are the solutions of the homogeneous equation. Such terms will give rise to secular (unbounded) terms in the solution $w_n(t,\tau)$. Since we have already assumed that the functions $y_0(x,t,\tau), y_1(x,t,\tau), \cdots$ are bounded on timescale of $O(\varepsilon^{-1})$. Thus, to have secular free behavior, the following solvability conditions are imposed in Eq. (26), that is,

$$A_{n0}'(\tau) + \frac{(n\pi)^2\delta}{2}A_{n0}(\tau) = 0, \\ B_{n0}'(\tau) + \frac{(n\pi)^2\delta}{2}B_{n0}(\tau) = 0. \tag{27}$$

The solutions to above system of two uncoupled ordinary differential equations in (27) are given as follows

$$A_{n0}(\tau) = A_{n0}(0)\mathrm{e}^{-\frac{(n\pi)^2\delta}{2}\tau}, \\ B_{n0}(\tau) = B_{n0}(0)\mathrm{e}^{-\frac{(n\pi)^2\delta}{2}\tau}, \tag{28}$$

where $A_{n0}(0)$and $B_{n0}(0)$ are given in Eqs. (18) and (19), respectively. Thus, by using Eq. (28) into Eq. (16), the complete solution to $O(1)$-problem is given as follows,

$$y_0(x,t,\tau) = \sum_{n=1}^{\infty}\mathrm{e}^{-\frac{(n\pi)^2\delta}{2}\tau}(A_{n0}(0)\cos(n\pi t) \\ + B_{n0}(0)\sin(n\pi t))\sin(n\pi x). \tag{29}$$

Now, by substituting $\tau = \varepsilon t$ into the expression, $-\frac{(n\pi)^2\delta}{2}\tau$, and then dividing the so-obtained expression by t, the damping parameter Γ_n for all the oscillation modes can be approximated by,

$$\Gamma_n = -\varepsilon\frac{(n\pi)^2\delta}{2}. \tag{30}$$

Thus, from Eq. (26) with Eq. (27), it follows that

$$w_{n_{tt}} + (n\pi)^2 w_n = \sum_{m=1,m\neq n}^{\infty} 4V(m\pi)^2 \left(A_{m0}(\tau)\sin(m\pi t) - B_{m0}(\tau)\cos(m\pi t)\right)\Theta_{mn}. \quad (31)$$

It can be seen that the Eq. (31) is a second order nonhomogeneous ordinary differential equation in $w_n(t,\tau)$, which yields a homogeneous solution to a homogeneous part of equation and a particular integral to a nonhomogeneous part of equation. Thus, the total solution to Eq. (31) is given as

$$w_n(t,\tau) = A_{n1}(\tau)\cos(n\pi t) + B_{n1}(\tau)\sin(n\pi t) + \sum_{m=1,m\neq n}^{\infty} \frac{4V(m\pi)^2\Theta_{mn}}{(n\pi)^2-(m\pi)^2}\left(A_{m0}(\tau)\cos(m\pi t) - B_{m0}(\tau)\sin(m\pi t)\right), \quad (32)$$

where $A_{n1}(\tau)$ and $B_{n1}(\tau)$ are yet undetermined functions of a slow variable τ, these functions can be obtained from the $O(\varepsilon^2)$-problem. Thus, the Eq. (20) with Eq. (32) can be expressed as

$$y_1(x,t,\tau) = \sum_{n=1}^{\infty}\left\{A_{n1}(\tau)\cos(n\pi t) + B_{n1}(\tau)\sin(n\pi t) + \sum_{m=1,m\neq n}^{\infty} \frac{4V(m\pi)^2\Theta_{mn}}{(n\pi)^2-(m\pi)^2}\left(A_{m0}(\tau)\cos(m\pi t) - B_{m0}(\tau)\sin(m\pi t)\right)\right\}\sin(n\pi x). \quad (33)$$

Now, by using the inner product (17) and the initial values in Eq. (15) with Eq. (16) into Eq. (33), it follows that $A_{n1}(0)$ and $B_{n1}(0)$ are given by

$$A_{n1}(0) = -\sum_{m=1,m\neq n}^{\infty} \frac{4V(m\pi)^2\Theta_{mn}}{(n\pi)^2-(m\pi)^2}A_{m0}(0), \quad (34)$$

$$n\pi B_{n1}(0) = \sum_{m=1,m\neq n}^{\infty} \frac{4V(m\pi)^3\Theta_{mn}}{(n\pi)^2-(m\pi)^2}B_{m0}(0) - A'_{n0}(0). \quad (35)$$

It can be observed that the solution $y_1(x,t,\tau)$ still contains infinitely many undetermined functions $A_{n1}(\tau)$ and $B_{n1}(\tau)$, for $n \in \mathbb{N}$. These unknown functions will be used to prevent the unbounded terms in solution $y_2(x,t,\tau)$. At this time, it is not reasonable to construct the higher order calculations. This is a reason, we can take $A_{n1}(\tau) = A_{n1}(0)$ and $B_{n1}(\tau) = B_{n1}(0)$. So far, a formal asymptotic expansion $y(x,t,\tau) = y_0(x,t,\tau) + \varepsilon y_1(x,t,\tau)$ has been constructed for $u(x,t)$, where $y_0(x,t,\tau)$ and $y_1(x,t,\tau)$ are continuously differentiable two times with respect to t, two times with respect to x, and infinitely many times with respect to τ.

RESULTS AND DISCUSSION

The aim of this section is to comment, interpret and explain the results obtained in previous section. Using the complete analytical solution of the $O(1)$-problem, the influence of small parameter ε and the damping parameter on the axially translating system will be discussed in detail. By using the $O(1)$- and the $O(\varepsilon)$-solutions, it turns out

$$u(x,t) = \sum_{n=1}^{\infty} \mathrm{e}^{-\frac{(n\pi)^2\delta}{2}\varepsilon t}\left(A_{n0}(0)\cos(n\pi t) + B_{n0}(0)\sin(n\pi t)\right)\sin(n\pi x),$$

where $A_{n0}(0)$ and $B_{n0}(0)$ are given by Eqs. (18) and (19). From physical view point, all terms can be explained in above solution to the IBVP (4)-(6). The terms $A_{n0}(0)\cos(n\pi t) + B_{n0}(0)\sin(n\pi t)$ are the oscillation terms obtained from a time-dependent part of the equation. These terms oscillate with frequencies n for $n \in N$, where sine terms have maximum oscillation amplitudes $A_{n0}(0)$ and cosine terms have maximum oscillation amplitudes $B_{n0}(0)$. The term $\mathrm{e}^{-\frac{(n\pi)^2\delta}{2}\varepsilon t}$ is arisen due to internal damping in the system. This term indicates that as the time parameter t will increase for fixed values of δ and ε the size of the oscillation amplitudes $A_{n0}(0)$ and $B_{n0}(0)$ will start to decrease and it also shown that as mode number n starts to increase the oscillation amplitudes tend to decrease for fixed δ, ε, and t. The last term $\sin(n\pi x)$ is the solution of the space-dependent part which describes the shapes of the oscillation curves along x-axis for fixed values of the time parameter t.

CONCLUSIONS AND FUTURE WORK

In this research article, an initial-boundary value problem (IBVP) for the internally damped axially translating continua has been studied. Solving the IBVP a method of two timescales has successfully been applied to obtain the analytic solutions of a proposed mathematical model. This mathematical model is based upon the transversal vibrations of a conveyor belt system. It has been shown, in this paper, that all oscillation modes are damped for the system. It has also been shown that damping rates are, in fact, depending on mode numbers n. From mechanical point of view, this response is reasonable because as oscillations increase the more heat is generated in the system which internally damps the vibratory energy of motion. As modes increase the oscillation amplitudes decrease and the belt system gets stable. This research problem can further be extended to internal damping of an axially translating beam with constant and time-dependent velocities.

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